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Routes to complete synchronization via phase synchronization in coupled nonidentical chaotic oscillators

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We study the transition route to complete synchronization through phase synchonization in generic coupled nonidentical chaotic oscillators. Through numerical studies, *two* routes are found, i.e., one, via lag synchronization, the other, via the intermittent chaotic burst state without lag synchronization. We claim that these two routes are universal. As evidence, we analyze several examples on the basis of *the conventional theory of intermittency* in the presence of noise.

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Synchronization phenomena in chaotic systems have attracted much attention since the work of Pecora and Carroll [1]. Recently, various types of chaos synchronization have been studied theoretically and observed experimentally in many different disciplines of science, e.g., laser systems [2], electronic circuits [3], chemical and biological systems [4], and secure communications [5]. Among the types of chaos synchronization, *phase synchronization* (PS) has become one of the most active fields of research since the first report of Rosenblum *et al.* [6].

The study of Rosenblum *et al.* focused on the PS of nonidentical chaotic oscillators and observed that the oscillators show a phase coherent rotation, i.e., the phase angle of the rotation increases steadily accompanied by chaotic fluctuations as time goes on. As the coupling strength is increased, a transition from a nonsynchronous state to a phase synchronous state occurs through an intermittently phase-locked state, where the phase difference between the two chaotic oscillators increases (or decreases) persistently with an intermittent sequence of 2π phase jumps [6–8]. They successively found that a relationship is established between the chaotic amplitudes of the oscillators as the coupling strength increases further. As a result, the states of two interacting systems coincide, if one is shifted in time. This new synchronous state is referred to as *lag synchronization* (LS) [9,10]. With a further increase of the coupling, the lag time decreases and the system asymptotically goes to *complete synchronization* (CS) [11]. These findings led Rosenblum *et al.* to propose the route from a nonsynchronous state to the CS state through successive transitions of PS and LS in two mutually coupled chaotic oscillators.

The motivation of our study is to address the following question: *Is the scenario of transitions to CS, via PS and LS, a universal route in the case of coupled chaotic oscillators with a parameter mismatch?* Through statistical analysis, in this paper, we present counterexamples to Rosenblum *et al.*'s route, and so show that a certain class of coupled nonidentical chaotic oscillators (chaotic oscillators with a slight parameter mismatch) follow a different transition route to CS. We will show that these two routes can be considered as universal on the basis of the conventional theory of intermittency in the presence of noise [12].

For the first example, we study coupled hyperchaotic Rössler oscillators (CHRO) whose equations are written as

$$\dot{x}_{1,2} = -\Omega_{1,2}\, y_{1,2} - z_{1,2} + \epsilon(x_{2,1} - x_{1,2}),$$

$$\dot{y}_{1,2} = \Omega_{1,2}\, x_{1,2} + 0.25\, y_{1,2} + w_{1,2},$$

$$\dot{z}_{1,2} = 3.0 + x_{1,2} z_{1,2},$$

$$\dot{w}_{1,2} = -0.5\, z_{1,2} + 0.05\, w_{1,2} + \epsilon(w_{2,1} - w_{1,2}), \tag{1}$$

where two variables x and w are mutually coupled [13]. Here $\Omega_{1,2} = 1.0 \pm \Delta\Omega/2$ are the overall frequencies of chaotic oscillators which are slightly detuned by $\Delta\Omega = 0.001$, and ϵ is the coupling strength. Since $y_{1,2}$ variables oscillate chaotically around a fixed center $y_0 = 0$, the phase can be defined around the center of the rotation $(v_0 = 0, y_0 = 0)$ by transforming $v_{1,2} \equiv \dot{y}_{1,2}/\Omega_{1,2}$ and $y_{1,2}$ into polar variables, i.e., $A_{1,2} = (v_{1,2}^2 + y_{1,2}^2)^{1/2}$ and $\theta_{1,2} = \tan^{-1}(y_{1,2}/v_{1,2})$ [14,15]. The phase equations of the system are expressed as follows:

$$\dot{\theta}_{1,2} = \Omega_{1,2} - \epsilon\left(\frac{x_{2,1} - x_{1,2}}{A_{1,2}} + \frac{w_{2,1} - w_{1,2}}{A_{1,2}\,\Omega_{1,2}}\right)\sin\theta_{1,2} + \left(\frac{z_{1,2}}{A_{1,2}} - \frac{\cos\theta_{1,2}}{4} + \frac{0.5 z_{1,2} + 0.05 w_{1,2}}{A_{1,2}\,\Omega_{1,2}}\right)\sin\theta_{1,2}. \tag{2}$$

Then, the phase difference between the two oscillators is given by $\phi = \theta_1 - \theta_2$.

We numerically study the dynamics of the phase difference ϕ as the coupling strength ϵ is varied. Figure 1 shows typical dynamic behaviors of the phase difference ϕ of the CHRO. For $\epsilon = 0.13$, the system shows $\pm 2\pi$ phase jumps with the intermittent chaotic bursts while their phases are

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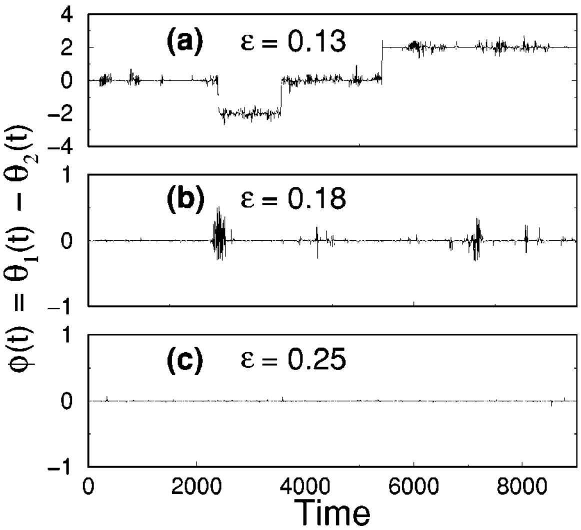

FIG. 1. Time series of the phase difference ϕ (measured in units of π) that show (a) intermittent $\pm 2\pi$ phase jumps, (b) PS state with chaotic intermittent bursts near the threshold of PS to CS, and (c) a state near CS.

locked. As we increase the coupling strength to $\epsilon = 0.18$, there are no phase jumps to be seen, although intermittent bursts still persist. Those bursts rarely appear when the coupling strength is further increased up to $\epsilon = 0.25$. The most notable difference in transition to CS between the CHRO and coupled Rössler oscillators is that *no phase shift is observed* when the phase of the CHRO is synchronized (Fig. 1). This implies there is no chance for LS to occur in case of the CHRO. On the other hand, in the case of coupled Rössler oscillators, $\pi/2$ phase shift is observed [6]. Other notable differences of the CHRO compared with coupled Rössler oscillators are as follows: (i) The phase jumps occur in both directions [Fig. 1(a)]. (ii) Near the transition threshold, large chaotic bursts are observed, instead of the LS state [Fig. 1(b)].

In order to understand these differences analytically, we need to study the phase difference dynamics more closely. From Eq. (2), the dynamic equation of the phase difference $\phi = \theta_1 - \theta_2$ can be written as

$$\frac{d\phi}{dt} = \Delta\Omega + \alpha \sin\phi + \beta \sin\frac{\phi}{2} + \gamma, \tag{3}$$

where

$$\alpha = -0.25\cos(\theta_1 + \theta_2),$$

$$\beta = \frac{2}{A_1}\cos\left(\frac{\theta_1+\theta_2}{2}\right)\left[z_1 + \frac{0.5z_1 + 0.05w_1}{\Omega_1} + \epsilon\left(x_1 - x_2 + \frac{w_1 - w_2}{\Omega_1}\right)\right],$$

$$\gamma = \left[\epsilon\left(\frac{x_1-x_2}{A_1} + \frac{w_1-w_2}{A_1\,\Omega_1} - \frac{x_2-x_1}{A_2} - \frac{w_2-w_1}{A_2\,\Omega_2}\right) + \frac{z_1}{A_1} - \frac{z_2}{A_2} + \frac{0.5z_1+0.05w_1}{A_1\,\Omega_1} - \frac{0.5z_2+0.05w_2}{A_2\,\Omega_2}\right]\sin\theta_2 .$$

It is well known that there are two different time scales in ϕ dynamics, i.e., the fast one $2\pi/\Omega_{1,2}$, which is related with the frequency of each individual oscillator in Eq. (2), and the slow one $2\pi/\Delta\Omega$, which is the characteristic time scale of ϕ dynamics originating from the parameter mismatch.

In our previous study [16,17], it was shown that if the term γ in Eq. (3) is absent, the time evolution of ϕ stops at $\phi^* \approx 0$ which is the stable fixed point. If we expand the equation around ϕ^*, Eq. (3) becomes $d\phi/dt = \Delta\Omega + A\phi - B\phi^3 + \gamma$ where $A = (\alpha + \beta/2)$ and $B = 1/3(\alpha + \beta/8)$. As the trajectory of ϕ nears the tangent point, ϕ varies very slowly (it can be regarded as constant) in comparison with $\theta_{1,2}$ and γ, which are fast varying terms. Therefore the above equation can be converted into a difference equation between the sectioning time τ, which is the average time for one revolution of one of the oscillators, in the following procedure.

$$d\phi = \tau\Delta\Omega + \phi\int_{n\tau}^{(n+1)\tau} A\,dt - \phi^3\int_{n\tau}^{(n+1)\tau} B\,dt + \int_{n\tau}^{(n+1)\tau}\gamma\,dt. \tag{4}$$

If we let $d\phi \approx \phi_{n+1} - \phi_n$, then the equation can be discretized as

$$\phi_{n+1} = a_n\phi_n - b_n\phi_n^3 + \xi_n , \tag{5}$$

where $a_n = 1 + \int_{n\tau}^{(n+1)\tau} A\,dt$, $b_n = \int_{n\tau}^{(n+1)\tau} B\,dt$, and $\xi_n = \int_{n\tau}^{(n+1)\tau}\gamma\,dt + \tau\Delta\Omega$. Notice that Eq. (5) is the equation of the local Poincaré map of Type-II intermittency [18] under the influence of noise if the fast dynamics of ϕ, i.e., ξ_n, can be regarded as an external noise. Equation (5) effectively becomes $\phi_{n+1} \approx a_n\phi_n + \xi_n$ near the transition since the value of ϕ_n is very small most of the time except during the bursts [Fig. 1(b)]. Then, the equation takes the form of on-off intermittency [19] in the presence of noise. That is, Eq. (5) predicts that the ϕ dynamics of the system transits to CS through on-off intermittency without LS. To study the transitional behavior more closely, we obtain the phase portraits of $y_1(t)$ versus $y_2(t)$, as shown in Figs. 2(a), 2(b), and 2(c), where ϕ exhibits phase jumps, intermittent bursts, and alomst CS, respectively.

Typically, we investigate the probability distribution of the average time interval between two consecutive bursts in $\phi(t)$, i.e., the average length of the laminar states since it is well known that on-off intermittency is usually observed before CS. Figure 2(f) shows the $-3/2$ slope for the shorter range of off-state and exponential drops for the longer range laminar phases. In between is the shoulder that connects these two curves. This is the hallmark of on-off intermittency in the presence of noise [20–23]. Therefore the nature of the intermittent bursts is indeed on-off intermittency. This, in

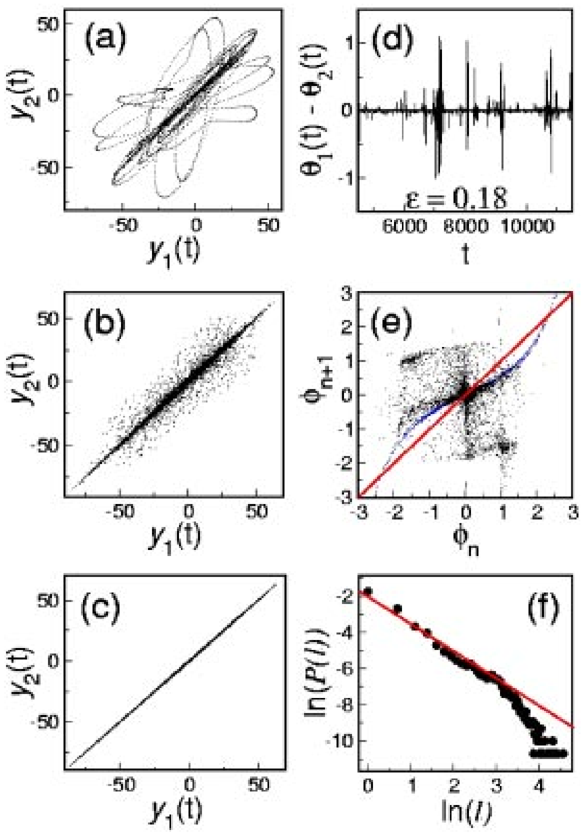

FIG. 2. The projection of the attractor of the CHRO in the plane $(y_1(t), y_2(t))$ for (a) $\epsilon = 0.05$ (nonsynchronized state), (b) $\epsilon = 0.18$ (PS state) and (c) $\epsilon = 0.25$ (almost CS state). Notice that large excursions from the diagonal line are observed for a nonsynchronized state. As the coupling strength increases, the points of (y_1, y_2) coalesces to the diagonal line statistically. (d) Phase difference of the CHRO in the PS regime $\epsilon = 0.18$. (e) The local Poincaré map that statistically follows a cubic curve and fuzzy line near center show attractor bubbling. (f) Probability distribution of the average laminar states shows the typical characteristic scaling exponent $-3/2$ and the shoulder.

turn, justifies our conjecture that *the ξ_n term, which comes from the fast dynamics of the ϕ, can be regarded as external noise*. So the observation of on-off intermittency supports our point of veiw that the phase dynamics of the CHRO can be treated as *Type-II intermittency in the presence of noise*.

For the second example, we study coupled Lorenz oscillators with a slight parameter mismatch in order to show the universal character of our observation. Coupled Lorenz oscillators are represented by the following equations:

$$\dot{x}_{1,2} = \alpha(y_{1,2} - x_{1,2}) + \epsilon(x_{2,1} - x_{1,2}),$$

$$\dot{y}_{1,2} = \beta_{1,2} x_{1,2} - y_{1,2} - x_{1,2} z_{1,2},$$

$$\dot{z}_{1,2} = \gamma z_{1,2} + x_{1,2} y_{1,2}, \tag{6}$$

where $\alpha = 10.0$, $\beta_{1,2} = 28.0 \pm 0.001$, $\gamma = 8/3$, and the strength of the coupling is given by the parameter ϵ. The phase is well defined for $(z, \dot{z})$ in our parameter regime. Then, the phase difference equation can be expressed similarly to that of the CHRO [Eq. (3)]. Figure 3 shows the results that are very similar to those of the CHRO. In particular, notice that the phase difference is zero when two Lorenz oscillators are nearly in the PS state [Fig. 3(a)]. This means there is no LS state in the course of the transition to CS. These indicate that the behaviors that were observed in the CHRO can be a generic characteristic of chaos synchronization for a certain class of chaotic oscillators.

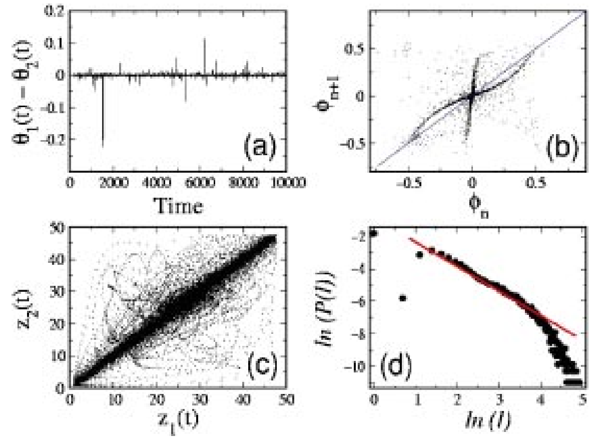

FIG. 3. (a) Phase difference of coupled Lorenz oscillators in PS regime $\epsilon = 4.5$. (b) The local Poincaré map shows a cubic curve (it is well fitted by $y = 0.478x + 2.001x^3$) and the almost vertical line in the center due to attractor bubbling. (c) Attractor bubbling in the variable z_1, z_2. (d) Probability distribution of average laminar lengths shows the typical characteristic scaling exponent $-3/2$ of on-off intermittency.

In the case of coupled Rössler oscillators [17,24], the phase difference equation, if it is expanded near the lag phase shift, is of the form $\phi_{n+1} = a_n \phi_n + b_n \phi_n^2 + \epsilon + \xi_n$,

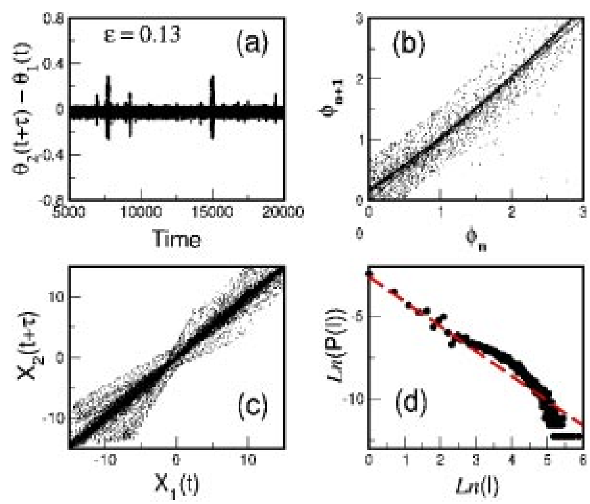

FIG. 4. (a) Phase difference of Rössler oscillators in LS regime $\epsilon = 0.13$, with lag time $\tau = 0.32$. (b) The local Poincaré map that statistically follows a quadratic curve. (c) Attractor bubbling in the variables $x_1(t), x_2(t+\tau)$. (d) Probability distribution of average laminar states shows the typical characteristic scaling exponent $-3/2$ of on-off intermittency with noise.

where ϵ and ξ_n are the channel width and the noise term from the fast dynamics of the system, respectively. Notice that the equation can be interpreted as the local Poincaré map of Type-I intermittency with noise. On-off intermittency also can be expected to occur by the same argument as is given previously. So, we investigate coupled Rössler oscillators at the onset of the LS transition. In this case, the phase difference of the two oscillators is locked at the lag phase (near $\pi/2$). This phase difference is maintained until the amplitudes are locked with a constant time delay, i.e., the LS state. The results are shown in Fig. 4. If we compensate the lag in phase, the phase difference also shows large bursts near the transition threshold just like the case of the CHRO [Fig. 4(a)]. As expected, the distribution of the average laminar states shows the characteristic behavior of on-off intermittency with noise [Fig. 4(d)]. Based on this observation, we may classify chaotic oscillators into two *universality classes* according to their transitional behavior to approach CS. This claim is reasonable for the following argument. Since the phase difference dynamics of generic nonidentical coupled oscillators can be divided into fast and slow motions, the quadratic (type-I intermittency [16,17]) and the cubic (type-II or III intermittencies [18,25]) structures in the local Poincaré map are the only *structurally stable* manifolds under the influence of external perturbations if we regard the fast motion as external noises. This is based on the well-known singularity theory [26].

In conclusion, we have shown an alternative route to CS through PS. This route, and the one proposed by Rosenblum *et al.*, may constitute two universal transition routes to CS through PS, in coupled chaotic oscillators with a parameter mismatch. The major difference of this proposed route is the absence of the LS state. Instead, only large intermitttent bursts are observed. Analytical study based on the conventional intermittency theory explains well the statistical behavior of these chaotic bursts, which demonstrated *on-off intermittency*. This observation supports our claim of the transition from PS to CS without LS.

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